European Research Studies Journal
Volume XXII, Issue 1, 2019

pp. 209-220

A Better Alternative to Conventional Bond in the Context of Risk Management

Submitted 11/2/19, 1st revision 18/2/19, accepted 25/2/19

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Abstract:

Under the assumption of flat spot curve, we define functional relationship between conventional and serial bonds' prices, when the bonds' parameters (par value, coupon rate and number of periods) are equal.

Furthermore, we conduct a thorough study of joint behavior of conventional and serial bonds' durations, which suggests that if spot curve is flat, and the bonds' parameters (coupon rate and number of periods) are equal, then conventional bond's durations (Macaulay and modified) significantly exceed serial bond's durations.

That is, all things being equal, conventional bond has considerably greater weighted average time until repayment and is much more exposed to interest rate risk than serial bond.

Keywords: *Conventional bond, serial bond, bond price, Macaulay duration, modified duration.*

JEL Classification: *C00, C61, G12, G32.*

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1. Introduction

The bonds are simplest fixed income securities. A bond is a debt instrument issued by a borrower, who is then required to repay to the lender the amount borrowed plus coupon payments (interest payments) over a fixed period. The most common type of bond is the conventional or bullet bond. This is a bond paying periodic fixed coupon payments over a specified period, with the return of par value of the bond on the maturity date. Nevertheless, like many securities, bonds have many different varieties, such as serial bonds, where the par value is paid back in equal installments. Therefore, serial bond's each payment is the sum of the installment and the interest rate on the outstanding debt.

Since serial bonds mature gradually over a period of years, they are used to finance projects that provide a consistent income stream for bond repayment. Most state and local governments issue their debt in the form of serial bond. Serial bonds, although prevalent as state or municipal issues, are not common corporation issues today (Guerard and Schwartz, 2007). However, corporations sometimes use serial bonds to meet a certain financing need.

By construction, conventional and serial bonds have different cash flow structures. Therefore, all things being equal, this essential distinction naturally causes differences between bonds' characteristics, such as prices, yields to maturity, durations, etc. From this perspective, we define functional relationship between conventional and serial bonds' prices, when spot curve is flat, and the bonds' parameters are equal. Furthermore, under the above-mentioned assumptions, we examine joint behavior of conventional and serial bonds' durations (Macaulay and modified) – that is, differences between bonds' durations. In fact, flat-shaped spot curve condition is a strict assumption, since there is no reason one should discount cash flows occurring on different dates with a unique rate (Martellini, Priaulet and Priaulet, 2003). Nevertheless, simplification may appear to be a convenient way to derive the main results of the paper. In subsequent research, we will generalize the findings of this paper.

Although there is abundant literature on all aspects related to bonds, a general look at the existing literature points toward a lack of studies that define relationships between the bonds' characteristics, when the bonds' parameters are equal. Even so, the findings of this research may be valuable for academics and practitioners. For instance, the comparative analysis of conventional and serial bonds' durations may contribute to risk management in the context of interest rate risk, which is the major risk faced by investors in the bond market (Fabozzi, 2007).

The remainder of this paper is organized as follows: Section 2 provides necessary definitions and assumptions, Sections 3 and 4 present the main findings, and Section 5 concludes.

2. Theoretical background

2.1 Preliminary definitions

2.1.1 Bond price

The price of a bond is the sum of the present value of all expected cash flows, hence it is given by

$$P(B(0,1),\dots,B(0,n),CF_1,\dots.,CF_n,n) = \sum_{t=1}^{n} B(0,t)CF_t,$$

where $B(0,t)$ is present value (time 0) of one unit of cash flow to be received at time t, CF_t is cash flow to be received at time t, n is number of periods, and $P(B(0,1),\dots,B(0,n),CF_1,\dots.,CF_n,n)$ is bond's fair price.

2.1.2 Spot rate and spot curve

Spot rate is the discount rate of a single future cash flow. A coupon bond can be viewed as a bundle of zero-coupon bonds, and the unbundled cash flows can be valued separately. All cash flows discounted with respective spot rates will sum-up to the bond's price. The discount factor $B(0,t)$ is defined as

$$B(0,t) = \frac{1}{\big(1+R(0,t)\big)^t},$$

where $R(0,t)$ is the spot rate (discount rate) at time 0 for an investment up to time t.

Spot rates as a function of maturity is called spot curve (Martellini *et al.*, 2003). The most typical shape for the spot curve is upward-sloping spot curve, which implies that the longer the maturity the higher the yield. A downward-sloping spot curve indicates that the longer the maturity, the lower the yield. The case when yield is approximately the same regardless of the maturity is referred to as a flat spot curve.

2.1.3 Yield to maturity

Yield to maturity is some weighted average of spot rates (Smith, 2011; CFA Institute Investment Series, 2015). In other words, yield to maturity is a single discount rate that equates the present value of a bond's cash flows to its market price.

$$P(R(0,1),\dots,R(0,n),CF_1,\dots,CF_n,n) = \sum_{t=1}^{n}\frac{CF_t}{\big(1+R(0,t)\big)^t} = \sum_{t=1}^{n}\frac{CF_t}{(1+y)^t} = P(y,CF_1,\dots,CF_n,n),$$

where y is yield to maturity.

2.1.4 Macaulay and modified durations

The Macaulay duration (Macaulay, 1938) of a bond is represented as the weighted average period of time until receipt of a bond's cash flows. Formally it is defined as

$$D(y,CF_1,\dots,CF_n,n) = -(1+y)\frac{\partial P(y,CF_1,\dots,CF_n,n)/P(y,CF_1,\dots,CF_n,n)}{\partial y},$$

where $D(y, CF_1, \dots, CF_n, n)$ is bond's Macaulay duration.

In contrast to Macaulay duration, modified duration is a measure of the approximate price sensitivity of a bond to yield changes. More precisely, modified duration is the approximate percentage change in bond's price for a 100-basis point change in yield, assuming that bond's expected cash flows do not change when the yield changes. Modified duration is defined as

$$MD(y, CF_1, \dots, CF_n, n) = \frac{D(y, CF_1, \dots, CF_n, n)}{1+y},$$

where $MD(y, CF_1, \dots, CF_n, n)$ is bond's modified duration.

2.2 Assumptions

The central assumption of this research is that conventional and serial bonds have equal parameters, in other words, conventional and serial bonds' par values, coupon rates and number of periods are equal. In addition, since the frequency of the coupon payments doesn't play a role in the context of this paper, therefore we assume for simplicity that conventional and serial bonds pay annual coupon payments.

For the sake of convenience, throughout this paper it is accepted that spot curve is flat-shaped. This assumption implies that conventional and serial bonds' yields to maturity are equal.

3. Relationship between conventional and serial bonds' prices

3.1 Theoretical results

As discussed earlier, conventional bond is paying a regular fixed coupon payment over a specified period to maturity, with the return of par value on the maturity date. Consequently, the cash flow structure of conventional bond presumes that $CF_t = \alpha N$ for $t = \overline{1, n-1}$ and $CF_n = \alpha N + N$. Therefore, the price of conventional bond that pays periodic coupon payment can be represented by the following formula

$$P'(y, N, \alpha, n) = \sum_{t=1}^{n} \frac{\alpha N}{(1+y)^t} + \frac{N}{(1+y)^n},$$

where N is par value, α is coupon rate, and $P'(y, N, \alpha, n)$ is conventional bond's fair price.

In contrast to conventional bond, serial bond's par value is paid back in equal installments, therefore the payment at a given payment date is the sum of the installment and the interest rate on the outstanding debt. As a consequence, the cash flow structure of serial bond presumes that $CF_t = \alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}$ for $t = \overline{1, n}$. The price of a serial bond that pays periodic coupon payment is expressed by the following formula:

$$P''(y,N,\alpha,n) = \sum_{t=1}^{n} \frac{\alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}}{(1+y)^t},$$

where $P''(y,N,\alpha,n)$ is serial bond's fair price.

Since we assume that spot curve is flat, then $R(0,1) = \cdots = R(0,n) = y$. Hence, conventional and serial bonds' yields to maturity are equal.

Theorem 1. If conventional and serial bonds' parameters (par value, coupon rate and number of periods) are equal and $y \neq 0$, then under flat spot curve condition, holds the following equality

$$P'(y,N,\alpha,n) = P''(y,N,\alpha,n) + \frac{N(\alpha - y)}{ny^2}\left(1 - \frac{1+ny}{(1+y)^n}\right). \tag{1}$$

Proof. If we assume that spot curve is flat, and the bonds' parameters (par value, coupon rate and number of periods) are equal, we must have

$$P'(y,N,\alpha,n) = P''(y,N,\alpha,n) + \frac{N}{(1+y)^n} + \frac{\alpha N}{n}\underbrace{\sum_{t=1}^{n}\frac{t-1}{(1+y)^t}}_{Q_n} - \frac{N}{n}\underbrace{\sum_{t=1}^{n}\frac{1}{(1+y)^t}}_{Z_n}. \tag{2}$$

Substituting the following expressions in equation (2)

$$Q_n = \frac{(1+y)^n - ny - 1}{y^2(1+y)^n}, \qquad Z_n = \frac{1}{y}\left(1 - \frac{1}{(1+y)^n}\right),$$

and doing straightforward algebraic manipulations, we obtain equation (1).

Theorem 2. If conventional and serial bonds' parameters (par value, coupon rate and number of periods) are equal and $y \neq 0$, then under flat spot curve condition, holds the following relationship

a) $P'(y,N,\alpha,n) = P''(y,N,\alpha,n) = N$, when $\alpha = y$,

b) $N < P''(y,N,\alpha,n) < P'(y,N,\alpha,n)$, when $\alpha > y$,

c) $P'(y,N,\alpha,n) < P''(y,N,\alpha,n) < N$, when $\alpha < y$.

Proof. According to Theorem 1

$$P'(y,N,\alpha,n) = P''(y,N,\alpha,n) + \Delta P(y,N,\alpha,n),$$

where $\Delta P(y,N,\alpha,n) = \frac{N(\alpha - y)}{ny^2}(1 - (1+ny)/(1+y)^n)$.

Since $y, N, \alpha > 0$, $n \geq 2$ and $1 - (1+ny)/(1+y)^n > 0$, therefore the sign of the function $\Delta P(y,N,\alpha,n)$ depends on the term $(\alpha - y)$.[2] As a consequence, we take into consideration the following three cases:

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[2] *It is accepted that* $n \geq 2$, *since* $n = 1$ *condition implies zero-coupon bond.*

1. If $\alpha = y$, then $\Delta P(y, N, \alpha, n) = 0$, and bonds are selling at par. Therefore, *(a)* holds.
2. If $\alpha > y$, then $\Delta P(y, N, \alpha, n) > 0$, and bonds are selling at premium. Therefore, *(b)* holds.
3. If $\alpha < y$, then $\Delta P(y, N, \alpha, n) < 0$, and bonds are selling at discount. Therefore, *(c)* holds.

3.2 Empirical results

In order to graphically demonstrate the theoretical results formulated by Theorem 2, for simplicity we consider the functions $P'(y, N, \alpha, n)$ and $P''(y, N, \alpha, n)$ quoted as percentage of par value for the fixed values of α and n.

As it is evident from Figure 1, the common patterns of the graphs are mathematically formulated in Theorem 2. Particularly, the point *(a)* of Theorem 2 corresponds to the intersection point of the curves, where $\alpha = y$. The point *(b)* of Theorem 2 corresponds to the left side of the intersection point, where $\alpha > y$. Similarly, the point *(c)* corresponds to the right side of the intersection point, where $\alpha < y$.

Figure 1. *The functions $P'(y, N, \alpha, n)$ and $P''(y, N, \alpha, n)$ quoted as percentage of par value for the fixed values of α and n.*

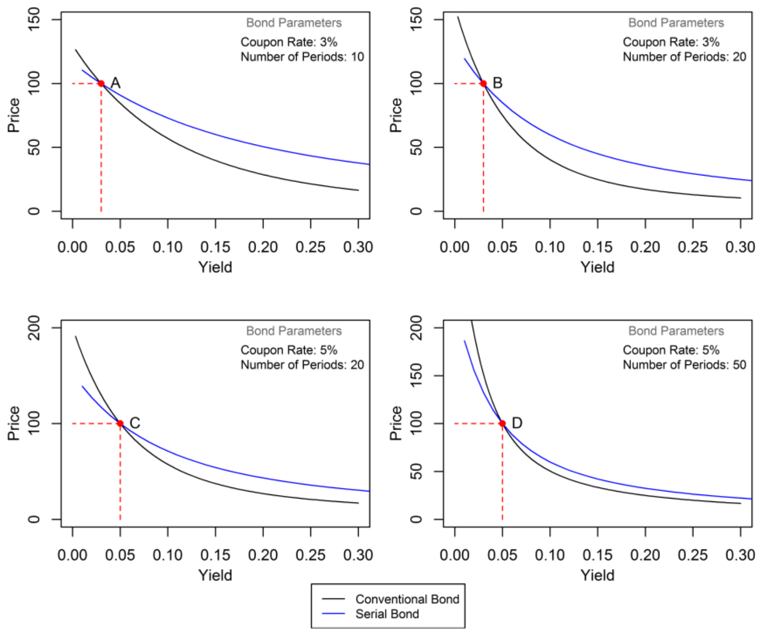


Table 1 shows local extreme values of the function $\Delta P(y, N, \alpha, n)$ quoted as percentage of par value. For illustrative purposes, the minimum and maximum values of the function $\Delta P(y, N, \alpha, n)$ over $\{(0;0.02]\times(0;0.02]\times\{20\}\}$ are -14.460 and 19.000, respectively (these extreme values are underlined in Table 1). This means that if

$y, \alpha \in (0; 0.02]$ and $n = 20$, then conventional bond's price is less than serial bond's price at most by 14.460 percent of par value and greater at most by 19.000 percent of par value, when spot curve is flat, and the bonds' parameters (par value, coupon rate and number of periods) are equal.

Table 1. *Local extreme values of the function $\Delta P(y, N, \alpha, n)$ quoted as percentage of par value*

Parameters			Minimum value	Maximum value
Coupon rate	Yield to maturity	Number of periods		
(0;0.001]	(0;0.001]	2	-0.050	0.050
(0;0.01]	(0;0.01]	10	-4.184	4.500
(0;0.02]	(0;0.02]	20	<u>-14.460</u>	<u>19.000</u>
(0;0.05]	(0;0.05]	50	-29.016	122.500
(0;0.1]	(0;0.1]	100	-29.428	495.000

As it is evident from Table 1, the difference between conventional and serial bonds' prices is significant. Furthermore, for some values of parameters, the difference between the bonds' prices is extremely large.

4. Relationship between conventional and serial bonds' durations

4.1 Theoretical results

In this section we discuss the relationship between conventional and serial bonds' durations (Macaulay and modified). From the definition of Macaulay duration follows that conventional and serial bonds' Macaulay durations are given by

$$D'(y, \alpha, n) = \left[\sum_{t=1}^{n} \frac{t\alpha N}{(1+y)^t} + \frac{nN}{(1+y)^n}\right] \Bigg/ \underbrace{\left[\sum_{t=1}^{n} \frac{\alpha N}{(1+y)^t} + \frac{N}{(1+y)^n}\right]}_{P'(y,N,\alpha,n)}, \qquad (3)$$

$$D''(y, \alpha, n) = \sum_{t=1}^{n} \frac{t\left(\alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}\right)}{(1+y)^t} \Bigg/ \underbrace{\sum_{t=1}^{n} \frac{\alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}}{(1+y)^t}}_{P''(y,N,\alpha,n)}, \qquad (4)$$

where $D'(y, \alpha, n)$ and $D''(y, \alpha, n)$ are conventional and serial bonds' Macaulay durations, respectively. Similarly, from the definition of modified duration follows that

$$MD'(y, \alpha, n) = \frac{D'(y, \alpha, n)}{1+y}, \qquad MD''(y, \alpha, n) = \frac{D''(y, \alpha, n)}{1+y},$$

where $MD'(y,\alpha,n)$ and $MD''(y,\alpha,n)$ are conventional and serial bonds' modified durations, respectively.

Theorem 3. If conventional and serial bonds' parameters (coupon rate and number of periods) are equal and $y \neq 0$, then under flat spot curve condition, holds the following relationship

$$D'(y,\alpha,n) > D''(y,\alpha,n), \text{ when } \alpha = y.$$

Proof. Assume that spot curve is flat, and the bonds' parameters (coupon rate and number of periods) are equal. If $\alpha = y$, then we can rewrite equations (3) and (4) as

$$D'(y,\alpha,n) = \sum_{t=1}^{n} \frac{ty}{(1+y)^t} + \frac{n}{(1+y)^n},$$

$$D''(y,\alpha,n) = \sum_{t=1}^{n} \frac{t\left(y\left(1-\frac{1}{n}(t-1)\right)+\frac{1}{n}\right)}{(1+y)^t}.$$

We only need to show that the left-hand side of the following equation is positive.

$$D'(y,\alpha,n) - D''(y,\alpha,n) = \frac{n}{(1+y)^n} + \frac{y}{n}\underbrace{\sum_{t=1}^{n}\frac{t(t-1)}{(1+y)^t}}_{M_n} - \frac{1}{n}\underbrace{\sum_{t=1}^{n}\frac{t}{(1+y)^t}}_{S_n}. \tag{5}$$

By replacing M_n and S_n with the following expressions

$$M_n = \frac{-n^2y^2 - ny^2 + 2y(1+y)^n + 2(1+y)^n - 2ny - 2y - 2}{y^3(1+y)^n},$$

$$S_n = \frac{(1+y)^n + y(1+y)^n - ny - y - 1}{y^2(1+y)^n},$$

and doing some straightforward algebra we can rewrite (5) as

$$D'(y,\alpha,n) - D''(y,\alpha,n) = \frac{1}{ny^2(1+y)^n}\left(-ny^2 + y(1+y)^n + (1+y)^n - ny - y - 1\right).$$

Since $n \geq 2$, then applying binomial formula it is trivial to verify that

$$-ny^2 + y(1+y)^n + (1+y)^n - ny - y - 1 > 0.$$

Thus, $D'(y,\alpha,n) - D''(y,\alpha,n) > 0$, when $\alpha = y$.

Corollary 1. If conventional and serial bonds' parameters (coupon rate and number of periods) are equal and $y \neq 0$, then under flat spot curve condition, holds the following relationship

$$MD'(y,\alpha,n) > MD''(y,\alpha,n), \text{ when } \alpha = y.$$

Proof. The proof immediately follows from the definition of modified duration and Theorem 3.

4.2 Empirical results

In order to determine the relationship between conventional and serial bonds' durations for the cases when $\alpha > y$ and $\alpha < y$, instead of analytical approach we apply numerical methods, since analytical approach may require lengthy and sophisticated algebraic manipulations. Therefore, by assuming that $y, \alpha \in (0,0.3]$ and $n \in [2,100]$, we define and examine the following function

$$\Delta D: X \to \mathbb{R}$$
$$X = (0,0.3] \times (0,0.3] \times \{[2,100] \cap \mathbb{Z}^{+}\}$$
$$(y, \alpha, n) \mapsto D'(y, \alpha, n) - D''(y, \alpha, n)$$

where

$$D'(y, \alpha, n) = \left[\sum_{t=1}^{n} \frac{t\alpha N}{(1+y)^t} + \frac{nN}{(1+y)^n}\right] \Bigg/ \left[\sum_{t=1}^{n} \frac{\alpha N}{(1+y)^t} + \frac{N}{(1+y)^n}\right],$$

$$D'(y, \alpha, n) = \sum_{t=1}^{n} \frac{t\left(\alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}\right)}{(1+y)^t} \Bigg/ \sum_{t=1}^{n} \frac{\alpha\left(N - \frac{N}{n}(t-1)\right) + \frac{N}{n}}{(1+y)^t}.$$

We also examine the difference between bonds' modified durations, and the corresponding function is defined as

$$\Delta MD: X \to \mathbb{R}$$
$$X = (0,0.3] \times (0,0.3] \times \{[2,100] \cap \mathbb{Z}^{+}\}$$
$$(y, \alpha, n) \mapsto MD'(y, \alpha, n) - MD''(y, \alpha, n)$$

where

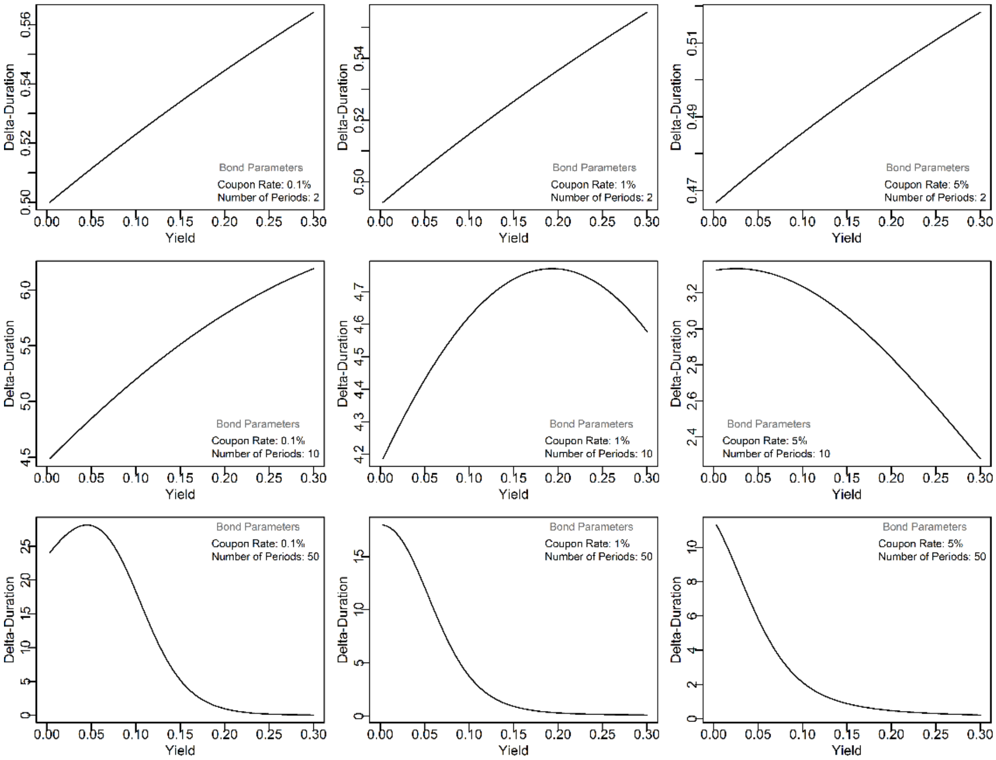
$$MD'(\alpha, y, n) = \frac{D'(\alpha, y, n)}{1+y}, \qquad MD''(\alpha, y, n) = \frac{D''(\alpha, y, n)}{1+y}.$$

It can be easily seen that the definition of domain X is impractical in the context of current financial market conditions, since the definition presumes that y, α and n attain extremely large values. Particularly, it is admitted that yield is bounded above by 0.3 (30 percent), but, for instance, according to U.S. Treasury yield curve the historical maximum yield for on-the-run securities is 0.0918 (9.18 percent) over the period of January 2, 1990 to September 18, 2018 (U.S. Department of the Treasury). Nevertheless, broadly defined domain X allows us to accept that X is maximal set of values for which the functions are defined. In other words, we accept that X is natural domain of the functions $\Delta D(y, \alpha, n)$ and $\Delta MD(y, \alpha, n)$.

The global minimum value of the function $\Delta D(y, \alpha, n)$ over domain X is positive (close to zero), consequently, the functions $\Delta D(y, \alpha, n)$ and $\Delta MD(y, \alpha, n)$ attain positive values. This means that conventional bond has greater Macaulay and modified durations than serial bond (including the cases when $\alpha > y$ and $\alpha < y$), when spot curve is flat, and the bonds' parameters (coupon rate and number of periods) are equal. In other words, conventional bond has greater weighted average time until repayment, and is more exposed to interest rate risk than serial bond.

The graphical representation of the function $\Delta D(y, \alpha, n)$ for the specific values of α and n is illustrated in Figure 2. Moreover, Figure 1A (Appendix 1) presents 3D plot of the function for the fixed values of n.

Figure 2. *The function $\Delta D(y, \alpha, n)$ for the fixed values of α and n.*



In order to determine whether the differences between bonds' durations (Macaulay and modified) are significant, we examine extreme values of the functions $\Delta D(y, \alpha, n)$ and $\Delta MD(y, \alpha, n)$ over subsets of domain X.

As Table 2 shows, conventional bond's Macaulay and modified durations significantly exceed serial bond's durations. For instance, the minimum and maximum values of the function $\Delta MD(y, \alpha, n)$ over {(0;0.02]×(0;0.02]×{20}} are 7.230 and 9.958, respectively (these extreme values are underlined in Table 2). This means that if $y, \alpha \in (0; 0.02]$ and $n = 20$, then conventional bond's modified duration is greater than serial bond's modified duration at least by 7.230 and at most by 9.958, when spot curve is flat, and the bonds' parameters (coupon rate and number of periods) are equal.

Table 2. *Local extreme values of the functions $\Delta D(y,\alpha,n)$ and $\Delta MD(y,\alpha,n)$*

Parameters			$\Delta D(y,\alpha,n)$		$\Delta MD(y,\alpha,n)$	
Coupon rate	Yield to maturity	Number of periods	Minimum value	Maximum value	Minimum value	Maximum value
(0;0.001]	(0;0.001]	2	0.499	0.500	0.499	0.500
(0;0.01]	(0;0.01]	10	4.169	4.582	4.169	4.537
(0;0.02]	(0;0.02]	20	7.375	10.157	<u>7.230</u>	<u>9.958</u>
(0;0.05]	(0;0.05]	50	5.836	33.777	5.558	32.168
(0;0.1]	(0;0.1]	100	0.473	89.007	0.430	80.920

Furthermore, as it is evident from Table 2, for some values of parameters, the differences between the bonds' durations (Macaulay and modified) are extremely large.

5. Conclusion

In this paper we defined functional relationship between conventional and serial bonds' prices, when spot curve is flat, and the bonds' parameters are equal. Under these assumptions, we also showed that conventional bonds' durations (Macaulay and modified) significantly exceed serial bonds' durations. Hence, all things being equal, conventional bond has considerably greater weighted average time until repayment and is much more exposed to interest rate risk than serial bond.

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Appendix 1

Figure 1A. *The function* $\Delta D(y, \alpha, n)$ *for the fixed values of* n