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Financial Crisis, Intervention and Performance Measurement

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Abstract:

Aspects of the financial markets that became apparent in the 2008 crisis were exacerbated by the intervention of monetary authorities. Financial markets under stress validate the general concept of Prospect Theory, under certain assumptions about the distributional characteristics of asset returns. This validation points to the need for re-examining performance metrics, such as the Sharpe Ratio and the Information Ratio. This analysis proposes new ratios that accommodate a higher moment of the portfolio return distribution. This alteration is reflected by the qualitative analysis of investment managers, which is performed by the performance evaluation industry, as it pertains to fixed income.

Keywords: *Quadratic Utility Function, Nonlinear Asset Response, Portfolio Kurtosis.*

JEL Classification: *G11, G15, C02*

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1. Introduction

The causes and effects of the 2007-08 financial crisis have been discussed extensively in academia, and anticipated largely, in the portfolio management industry. That, in spite of the fact that ingredients for the 'perfect storm' may appear to be nowhere near the levels of that period (Thalassinos *et al.*, 2014). At the same time, the crisis seems to have exerted a profound effect on the strategic approach of portfolio managers, which brings into sharp focus what was already known as a bifurcation of returns into those generated from normal market volatility, and the ones that result in bets on abrupt market events. The fact that responses of asset returns to indices introduce kurtosis in portfolio returns has already been researched, however, in the current market of low rates and low volatility, there is some preoccupation with tail-risk, which entails modeling the exposure to third and fourth moment deviations around the mean return. Witness the growth in portfolios marketed to both retail clients and institutional investors as deviating from a stated benchmark, in both the magnitude of weight changes across an investable universe (unconstrained strategies), and the length of holding period (strategies that feature a pronounced 'buy-and-hold' component). Often, strategies that deviate from the established indices that are based on market capitalization necessitate the construction of risk-based (risk-parity) benchmarks, a rather complicated prospect. It may be argued that recent developments are the result of central bank intervention (extensive monetary expansion and quantitative easing/tapering), which have been imposed on markets in the U.S. and Europe at different points in time. However, portfolio good performance measurement can still be achieved through a simple extension of quadratic utility optimization, under a set of non-restrictive assumptions. It is no longer valid to presume that the investor neglects higher moments of the return distribution, in arriving at optimal weights. At the same time, a very simple rule may offer insight into the sources of performance relative to any benchmark, if the effect of fat tails (kurtosis) is explicitly incorporated. For quadratic utility optimizers, kurtosis aversion is viewed as either platykurtosis-seeking or leptokurtosis-aversion. The investor observes kurtosis and operates at a 'prudent' trade-off between it and variance. Given the investment horizon, this trade-off leads to abrupt adjustments. A 'combined' risk tolerance captures this response as weights are adjusted in comparison to deviations of mean-variance portfolio returns from normality. The comparison of the probability of outperformance for a specific strategy, to its generation of returns through the operation at the prudent point, adheres to the central idea of Prospect Theory. In re-stating performance measurement metrics (Sharpe Ratio, Information Ratio) based on this preference reversal toward fat tails, one arrives at better judgment, concerning the probability of a strategy in outperforming its peers.

2. Higher Order Return Moments and Investor Utility

Portfolio returns that are anticipated, linear in some index or normally distributed, and those that are unanticipated, nonlinear in an index and fat-tailed, are related to each other. This study examines the behavior of higher order utility maximizing investors, who face nonlinear responses of strategy returns to current and lagged interest rate movements. I use a smooth utility function that is not strictly concave. This function is initially concave in returns, justifying the local solution. In a second stage, large deviations from portfolio normality trigger an investor reaction toward the resulting kurtosis in returns, in a predictable manner. Past experience or investment horizons prevent infinite positions. The proposed methodology is applied to retail ('40-Act) fixed income funds, but may have strong implications for institutional investors.

2.1 Violations in the Axiomatic Utility Formulation

For over half a century the optimization in Markowitz (1952) and Sharpe (1963) constituted the standard, in portfolio choice. Asset allocations depended on the expected return (mean) and risk (covariance) of assets. This idea was based on the assumptions that (i) return distributions possessed spherical symmetry, or (ii) investors were indifferent to higher moments of portfolio return distributions. Early extensions to this approach pointed out that, if utility was an *n*-th degree polynomial, moments as high as the *n*-th should be considered (Richter 1960). Samuelson (1967) had added to the controversy on the moments of portfolio wealth distribution by stressing the usefulness of mean and variance in situations that involved less risk, what he called 'compact' probabilities. Borch (1969) maintained that utility $u(x)$ was a polynomial for which the requirements $u'(x) > 0$ and $u''(x) < 0$ were satisfied only in certain intervals, an idea that is intimately related to the utility function proposed in the present study. The precise mechanism of deviation from quadratic utility was not apparent in these approaches. I attempt to investigate that mechanism.

Earlier studies also differentiate uncertainty from risk, challenging classic mean-variance utility formulation.[3] Kreps and Porteus (1978) portray investors who are not indifferent to the time of uncertainty resolution. Kahneman and Tversky's (1979) 'certainty effect' describes investors as risk-averse when exhibiting preference for sure gains over merely probably larger gains; and risk-seeking when exhibiting preference for merely probable losses over smaller losses that are certain. In Jensen and Donaldson (1985), time-consistent planning in utility-maximizing decisions remain ideally the same, as the resolution of uncertainty takes place over time. The quest should thus be finding a utility representation of preference ordering that guarantees consistent planning. Machina (1987) treats the psychological observation of a 'preference reversal' as one of the strongest challenges against the VNM axiomatic theory of choice under uncertainty. Kimball (1990) coins the term 'prudence' to capture investor reactions to the actual occurrence of uncertain events. The skewness of utility functions governs this precautionary motive, while variance only captures risk aversion. Liu and Longstaff (2003) fuse the effect of volatility changes into portfolios that differ for large and small returns. Investors hedge price movements with volatility jumps via a static buy-and-hold component. Such hedging behavior between linear effects and nonlinear changes is adopted in this analysis. Several studies link asset kurtosis to abrupt reactions toward extreme events.[4] Uncertainty makes risk-averse investors dislike kurtosis (Scott and Horvath, 1980). Investors that mistakenly assume normality, when some asset return distributions are leptokurtic, substantially underestimate the effects of uncertainty on their portfolio depending on the length of their investment horizon. The functional form of utility should capture investor preferences on the shape of the asset return distribution. This argument is traced back to literature that challenged axiomatic utility and mean-variance portfolio optimization. In my analysis, resolution of uncertainty manifested as kurtosis in expected portfolio returns, occurs at each rolling-sample. Each time, the investor chooses to either adjust portfolio weights or not, after having maximized portfolio returns by mean-variance. With no such previous knowledge, the investor should instantaneously make this adjustment, prompted by kurtosis in portfolio returns. Kahneman and Tversky's positive (negative) domain of the certainty effect would then pertain to the absence (presence) of kurtosis in portfolio returns, after applying weights obtained through the classic quadratic utility maximization. When quadratic portfolio returns are not kurtotic, (positive domain of outcomes concerning investor wealth), the investor does not undergo a secondary adjustment in portfolio weights, preferring a more secure gain. When they are kurtotic, (negative domain of outcomes concerning investor wealth), the investor goes through the secondary adjustment in portfolio weights, risking an uncertain loss, up to the point of a prudent adjustment. In the literature, there is concrete evidence of such local risk seeking behavior. This present study merely proposes a simple method for the programmatic incorporation of such apparent risk seeking, in the investor's effort to reduce overall risk through a 'hedging' procedure between variance risk and kurtosis uncertainty. In this framework the prudent investor exhibits consistent planning, given the proposed utility function. Interestingly, the investor also reverses attitudes toward uncertainty when kurtosis in portfolio returns is present. This preference reversal leads to the point of prudence where aversion to kurtosis compensates for risk from variance. Prudence as a concept is precautionary, as the investor responds to variance in an attempt to stonewall total risk, and thus maximize expected utility.[5]

2.2 Portfolio Strategy Returns and Quadratic Utility Weights

I postulate that the investor is variance-averse, in the constant relative risk aversion sense. However, the investor exhibits a selective response to deviations of quadratic portfolio returns from normality. This study explores some adjustments to quadratic weights, which would have taken place at the 2007-2008 financial crises. Through an assumption about the 'prudent' magnitude of adjustment, one observes the abrupt reaction and entrenchment of kurtosis, as only consistent deviations from normality that would induce portfolio adjustments. It is shown that operating at the proposed prudent point generally affects portfolio returns and value. The succession of abrupt reactions by periods of entrenchment in non-benchmark-constrained or buy-and-hold portions of wealth sustain profits not evident to just mean-variance investors:

$$\underset{\text{w/ respect to q}}{\text{maximize}} : E\left[U\left(r_p\right)\right] = \mathbf{q}^T E[\mathbf{x} \cdot \mathbf{b}] - \frac{1}{2 \cdot \lambda} \cdot \mathbf{q}^T \Omega \, \mathbf{q} \tag{1}$$

$$\text{subject to} : \mathbf{q}^T \cdot \mathbf{1} = 1, \text{ where } \mathbf{1} \text{ is a vector of 1's.}$$

$$\text{and} : -1 \le q_j \le 1$$

Investors find optimal weighs **q** based on estimates of expected returns [**x** · **b]** and covariance matrix Ω. The variance-averse investor has tolerance $\lambda>0$, and preference $-1/(2\lambda)<0$. The opposite is true for a variance seeker: $\lambda<0$, and $-1/(2\lambda)>0$. Negative preference toward variance increases when positive tolerance increases. Conversely, positive preference toward variance decreases when negative tolerance decreases. This is the classic formulation of quadratic utility maximization with respect to weights **q**, with tolerance λ toward variance. It implicitly assumes that kurtosis is zero. In this analysis, however, the vector **q** is adjusted for return kurtosis. This new subjective probability distribution is described through the parameters of mean, variance and kurtosis, only. Markets fluctuate within a range of mean-variance return optimization, if (1) above holds. When portfolio returns deviate from normality the investor adjusts weights in a way that amplifies or diminishes portfolio returns based on his combined tolerance toward risk from both the variance and kurtosis. Skewness in either direction is assumed to not affect utility. During abrupt market movements, fat tails overpower any change in skew of portfolio returns, from positive to negative as is often the case. The adjustment in portfolio weights captures all of these effects. I am applying this methodology to portfolio outperformance, relative to a benchmark. In that setting, 'long' and 'short' positions are defined in relation to benchmark weights, in which case skew could symmetrically appear on either side of a zero-sum weight position. I am only interested in exposure sources that are mean preserving, such as variance, and kurtosis. Recent developments in the application of extreme value theory, specifically model the left tail of the return distribution, effectively neglecting the effect on the average return. The desirability of these methodologies notwithstanding, the above is but a simple way of expressing utility of wealth that, in addition to preserving the mean of the return distribution, requires no inversion of co-skewness, or co-kurtosis matrices. Portfolio risk can then be approximated by the product of variance and kurtosis. Members of the same peer group can be compared based on that product, even as their approach varies. In the section below, I derive this product, from standard theory on utility of wealth.

2.3 Augmentation of Quadratic Utility that Accommodates Kurtosis

A fourth power utility function is assumed, $U(r_p)=b_0+b_1r_p+b_2r_p^2+b_3r_p^3+b_4r_p^4$. Taylor series expansion with $b_0=b_3=0$ yields relations (2) and (3). In addition to mean and variance, utility depends on θ_p^4, the normalized kurtosis of returns, in (4). Similar to risk coefficient λ, a tolerance toward kurtotic portfolio returns, ν, is introduced.

$$U(r_p) = U\left(\bar{r}_p\right) + U'\left(\bar{r}_p\right)\cdot\left(r_p - \bar{r}_p\right) + \frac{1}{2}\cdot U''\left(\bar{r}_p\right)\cdot\left(r_p - \bar{r}_p\right)^2 + \frac{1}{6}\cdot U'''\left(\bar{r}_p\right)\cdot\left(r_p - \bar{r}_p\right)^3 + $$

$$+\frac{1}{24}\cdot U^{(4)}\left(\bar{r}_p\right)\cdot\left(r_p - \bar{r}_p\right)^4 + H_5, \quad \text{with} \quad H_5 = 0. \qquad (2)$$

$$E\left[U(r_p)\right] = U\left(\bar{r}_p\right) + U'\left(\bar{r}_p\right)\cdot 0 + \frac{1}{2}U''\left(\bar{r}_p\right)\sigma_p^2 + \frac{1}{6}U'''\left(\bar{r}_p\right)\cdot 0 + \frac{1}{24}U^{(4)}\left(\bar{r}_p\right)\theta_p^4\left(\sigma_p^2\right)^2 =$$

$$= b_1\bar{r}_p + b_2\bar{r}_p^2 + b_4\bar{r}_p^4 + \left(b_2 + 6b_4\bar{r}_p^2\right)\sigma_p^2 + b_4\theta_p^4\left(\sigma_p^2\right)^2 = f\left(\bar{r}_p, \sigma_p^2, \theta_p^4\right) \qquad (3)$$

$$U\left(\bar{r}_p\right) = b_1\bar{r}_p + b_2\bar{r}_p^2 + b_4\bar{r}_p^4 \quad ; \quad U''\left(\bar{r}_p\right) = 2b_2 + 12b_4\bar{r}_p^2 \quad ; \quad U^{(4)}\left(\bar{r}_p\right) = 24b_4$$

$$E\left(r_p - \bar{r}_p\right) = 0, E\left(r_p - \bar{r}_p\right)^2 \equiv \sigma_p^2, \theta_p^3 \equiv E\left(r_p - \bar{r}_p\right)^3 = 0, \theta_p^4 \equiv E\left(r_p - \bar{r}_p\right)^4 \Big/ \left(\sigma_p^2\right)^2$$

$$\underset{\text{w/ respect to } \mathbf{q}}{\text{maximize}} : E\left[U(r_p)\right] = \mathbf{q}^T E[\mathbf{x}\cdot\mathbf{b}] - \frac{1}{2\cdot\lambda}\cdot\mathbf{q}^T\Omega\,\mathbf{q} - \frac{\left(\theta_p^4 - 3\right)}{4\cdot v}\left(\mathbf{q}^T\Omega\,\mathbf{q}\right)^2 \qquad (4)$$

$$\text{subject to}: \mathbf{q}^T\cdot\mathbf{1} = 1, \text{ where } \mathbf{1} \text{ is a vector of 1's.}$$

$$\text{and}: -1 \le q_j \le 1$$

Risk-averse investors should prefer no leptokurtosis ($\theta_p^4>3$) and would also perceive platykurtosis ($\theta_p^4<3$) as risk-mitigating: $v<0$ when $\theta_p^4<3$ and $v>0$ when $\theta_p^4>3$. That makes the preference toward kurtosis $-(\theta_p^4-3)/4v$ negative. The opposite is true for risk seekers: $v>0$ when $\theta_p^4<3$ and $v<0$ when $\theta_p^4>3$, while $-(\theta_p^4-3)/4v$ is positive. In kurtosis aversion, the negative preference improves with the absolute value of v, and vice versa. As $|v|$ goes up the investor seeks less risky, or platykurtotic portfolio returns and avoids the riskier leptokurtotic returns. In kurtosis seeking, the positive preference toward kurtosis deteriorates as $|v|$ increases, and vice versa. Risk seeking makes v positive in platykurtosis and negative in leptokurtosis. In this analysis, quadratic utility maximization in (1) is a stage of (4), with $\theta_p^4=3$. The **q**-weights in relation (4) come from (1), and can affect utility in a 'hedging' relation. The investor adjusts **q** to the fact that $\theta_p^4\neq3$, invoking a combined tolerance toward both variance and kurtosis, which is derived while optimizing utility with respect to weights **q**.[6]

2.4 Combined Tolerance toward Both Variance and Risk

The combined tolerance toward both moments exerts amplification or dampening effects of weights **q**, after kurtosis is observed. Weights are determined in two stages: (i) investors assume normal returns and derive **q**; (ii) investors observe portfolio kurtosis and adjust **q**, based on combined risk tolerance. In the second stage, the investor views portfolio kurtosis as a parameter whose value is estimated in the first. Equation (4) is restated as (5), where utility becomes a quadratic form in variance $m=\sigma_p^2$. The polynomial utility (4) becomes quadratic in variance of wealth, in contrast to the quadratic function of wealth. The first and second order conditions in (6) imply that the optimal point is a minimum when investors are variance-averse *($\lambda>0$)* and a maximum when they are variance seeking *($\lambda<0$)*.

$$E\left[U\left(r_p\right)\right]=am^2+bm+c;\text{ where } a=-\frac{\theta_p^4-3}{4\nu},\quad b=-\frac{1}{2\lambda},\quad c=\bar{r}_p,\quad m=\sigma_p^2 \qquad (5)$$

$$\text{FOC}:\frac{\partial E\left[U\left(r_p\right)\right]}{\partial m}=2am+b=0\Leftrightarrow\sigma_p^2\left(\theta_p^4-3\right)=-\frac{\nu}{\lambda} \qquad (6)$$

$$\text{SOC}:\frac{\partial^2 E\left[U\left(r_p\right)\right]}{\partial m^2}=2a\geq 0\Leftrightarrow\begin{Bmatrix}\nu\leq 0 & \text{when} & \theta_p^4\geq 3\\ \nu\geq 0 & \text{when} & \theta_p^4\leq 3\end{Bmatrix}\quad\text{(hedging)}$$

$$\lambda^{-1}+\nu^{-1}\left(\theta_p^4-3\right)\sigma_p^2=0\Rightarrow\xi\equiv\left[\lambda^{-1}+\nu^{-1}\left(\theta_p^4-3\right)\sigma_p^2\right]^{-1}\to\infty \qquad (7)$$

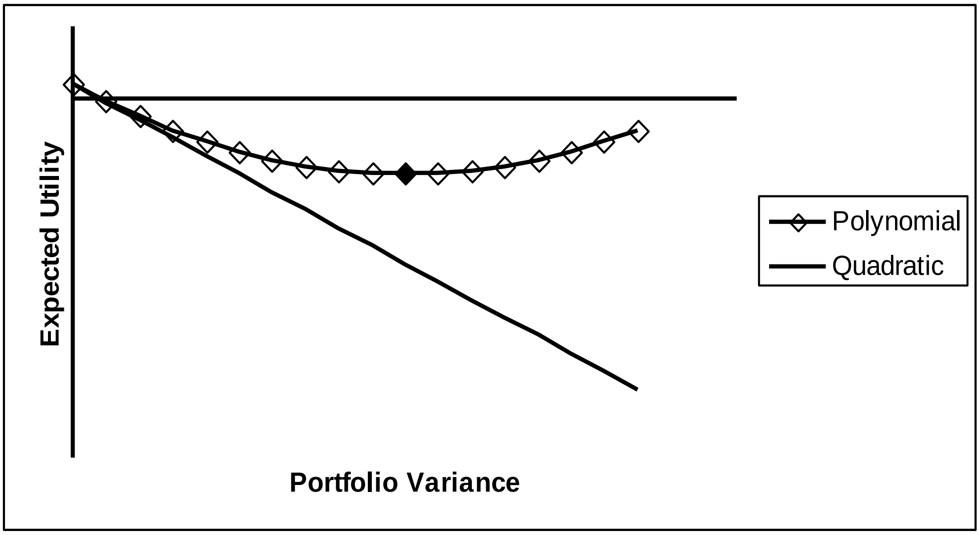

Figure 1. Prudent Point in Polynomial Utility Maximization

The 2008 financial crisis and subsequent central bank intervention promulgated through fund management the operation at a point represented by the darkened dot in Figure 1. Specifically, quadratic utility (1) is penalized as portfolio variance goes up, although polynomial utility (4) possesses a minimum beyond which volatility enhances utility. This minimum in polynomial utility is higher than quadratic utility by $\sigma_p^2/4\lambda$ and is a 'prudent point' in the following sense: the fund manager engages in a process that maximizes the vertical distance between a straight line of negative slope for mean-variance in (1) and the curved line in (4). Up to the darkened point, portfolio variance reduces utility in both the quadratic and polynomial functions (no retail investor or plan sponsor would agree that increasing variance raises utility). A prudent investor would not seek to raise portfolio variance beyond the prudent point, since the additional variance adds to polynomial utility indefinitely, contradicting his professed risk-aversion stance. When the investor 'hedges' variance-based returns with tail-based returns, his preference reversal can change him from variance-averse into variance-seeking, if he is not prudent. But, even operating at the prudent point, involves pressures exerted on market liquidity as portfolio weights approach infinity. In (7) above, combined tolerance for variance and kurtosis, ξ, may become infinite, straining market liquidity, as the investor gets 'surprised' by portfolio kurtosis.

Prudent condition (7) appears during the optimization of (4) with respect to **q**, and captures hedging behavior between risks related to portfolio return variance and uncertainty captured in kurtosis (preference reversal). In the case of leptokurtosis, a variance-averse investor is also prudently kurtosis seeking *($\lambda>0$, $\nu<0$)*, in (8). When variance seeking, the same investor is leptokurtosis-averse *($\lambda<0$, $\nu>0$)*, although this case has less applicability. In platykurtosis the reverse is true. When variance-averse, a maximizing investor is kurtosis-averse. When variance-seeking he is kurtosis seeking, in (9). Indicator variable $I_{[\,]}$ illustrates preference toward kurtosis, in (10).

$$\theta_p^4 > 3 \Rightarrow \left(\theta_p^4 - 3\right)\sigma_p^2 > 0 \Rightarrow -\frac{\nu}{\lambda} > 0 \Rightarrow \begin{Bmatrix} \nu < 0 \Rightarrow \nu = -|\nu_0|, & \lambda > 0, & |\nu_0| > 0 \\ \nu > 0 \Rightarrow \nu = +|\nu_0|, & \lambda < 0, & |\nu_0| > 0 \end{Bmatrix} \tag{8}$$

$$\theta_p^4 < 3 \Rightarrow \left(\theta_p^4 - 3\right)\sigma_p^2 < 0 \Rightarrow -\frac{\nu}{\lambda} < 0 \Rightarrow \begin{Bmatrix} \nu > 0 \Rightarrow \nu = +|\nu_0|, & \lambda > 0, & |\nu_0| > 0 \\ \nu < 0 \Rightarrow \nu = -|\nu_0|, & \lambda < 0, & |\nu_0| > 0 \end{Bmatrix} \tag{9}$$

$$\nu = -|\nu_0| \cdot I_{[\theta_p^4 > 3]} + |\nu_0| \cdot I_{[\theta_p^4 < 3]} \tag{10}$$

$$\pi_p \equiv \left(\theta_p^4 - 3\right)\sigma_p^2 = -\frac{\nu}{\lambda} \quad ; \quad \eta_p = \frac{\partial\left(\theta_p^4 - 3\right)}{\partial \sigma_p^2} \cdot \frac{\sigma_p^2}{\left(\theta_p^4 - 3\right)} = \frac{-\pi_p}{\sigma_p^2} \cdot \frac{1}{\left(\theta_p^4 - 3\right)} = -1 \tag{11}$$

$$\left(\theta_p^4 - 3\right)\nu + \overset{(+)}{h} = 0 \Rightarrow \nu = \frac{-h}{\left(\theta_p^4 - 3\right)}; \quad \theta_p^4 - 3 = -\frac{\nu}{\lambda}\frac{1}{\sigma_p^2} \Rightarrow \sigma_p^2 = h \cdot \underbrace{\frac{1}{\lambda\left(\theta_p^4 - 3\right)^2}}_{(>0)} \tag{12}$$

Portfolio variance can be thought of as the price of normalized kurtosis since the two concepts are inversely related in hedging. The product of excess kurtosis, times variance, π_p, defines 'iso-risk' combinations for which the ratio of tolerances stays the same, shown in elasticity condition (11). For variance aversion λ, the investor exhibits a unitarily elastic demand for kurtosis that reveals indifference to combinations of two sources of risk (η_p= - 1). The second order condition (12) shows that the greater the tendency to hedge between variance and kurtosis, *h*, the larger the portfolio variance. The prudent utility-maximizer views variance and kurtosis in portfolio returns as inversely (directly) related when his ratio of tolerances ν/λ is negative (positive). A variance-averse investor has a negative (positive) kurtosis tolerance toward lepto- (platy-) kurtotic portfolio returns. He assumes $\theta_p^4=3$ when maximizing utility, and is surprised to find that $\theta_p^4\neq 3$. To avoid contradicting the apparent preferences toward variance only and still account for kurtosis in utility, the investor augments combined tolerance so that first order condition (7) is approached, without leading to infinities. The procedure presented below relies on information on kurtosis tolerance that helps avoid infinite weights. Combined tolerance ξ exerts a certain effect on returns stemming from quadratic utility, which amounts to an adjustment of weights from **q** to **w.** For purposes of exposition, unconstrained utility maximization of (1) and (4) below, illustrates the effects of kurtosis on the optimal portfolio weights. The argumentation is then carried through to computational procedures used to arrive at results in this analysis.

3. Computational Procedure

Any two assets could help illustrate the maximization of polynomial utility, with respect to adjusted weights $\mathbf{w} = [w_1, w_2]$, only. This example can be extended for many assets, easily. The investor uses quadratic utility, arriving at unadjusted portfolio weights $\mathbf{q} = [q_1, q_2]$. Then, she compares portfolio kurtosis from this stage against the benchmark implied by her tolerance and adjusts weights to **w**, based on information. In the first order conditions, kurtosis is assumed equal to that of a normal distribution $(\theta_p^4=3)$, resulting in the quadratic utility weights **q,** in (14). In the case where the return distribution is normal $(\theta_p^4=3)$, weights **q** maximize polynomial utility, in (13).

$$E[U(r_p)] = w_1E[r_1] + w_2E[r_2] - \frac{1}{2\lambda}\left(w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2\right) - \\ - \frac{\theta_p^4 - 3}{4\nu}\left(w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2\right)^2 \qquad (13)$$

$$\begin{bmatrix} \dfrac{\partial E[U(r_p)]}{\partial q_1} \\ \dfrac{\partial E[U(r_p)]}{\partial q_2} \end{bmatrix} = \begin{bmatrix} E(r_1) \\ E(r_2) \end{bmatrix} - \lambda^{-1}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \nu^{-1}(3-3)\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\sigma_p^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow [q_1 \quad q_2] = \lambda\,[E(r_1) \quad E(r_2)]\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}^{-1} \Rightarrow \mathbf{q}^T = \lambda\, E[\mathbf{x}\cdot\mathbf{b}]\Omega^{-1} \qquad (14)$$

$$n \cdot \left(\frac{\theta_p^3}{6} + \frac{\theta_p^4 - 3}{24} \right) \sim \chi^2(2) \tag{15}$$

$$\begin{bmatrix} \dfrac{\partial E\left[U\left(r_p\right)\right]}{\partial w_1} \\ \dfrac{\partial E\left[U\left(r_p\right)\right]}{\partial w_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{w}^T = \xi \, E\left[\mathbf{x} \cdot \mathbf{b}\right]^T \, \Omega^{-1} \tag{16}$$

$$\xi = \left[\lambda^{-1} + \nu^{-1}\left(\theta_p^4 - 3\right)\sigma_p^2\right]^{-1} = \frac{\lambda \nu}{\nu + \lambda\left(\theta_p^4 - 3\right)\sigma_p^2} \equiv \frac{\lambda}{1 - \dfrac{\lambda\left(\theta_p^4 - 3\right)\sigma_p^2}{\left|\nu_0\right| I_{\left[\theta_p^4 > 3\right]} - \left|\nu_0\right| I_{\left[\theta_p^4 < 3\right]}}} \tag{17}$$

The first stage is used to test for kurtosis in portfolio returns and to set the indicator variables $I_{[\ ]}$ in (17). Test statistic (15) with θ_p^3 assumed zero determines if distributions deviate from normality, assuming they are not skewed (Bowman-Sheldon Test of Normality). With parameters θ_p^4 and σ_p^2 from stage 1 the first order condition of the second stage provides utility maximizing weights, in (16). The tolerance toward variance and kurtosis based on information $|\nu_0|$ implies non-infinite portfolio weights in (17). With this method, simple utility-optimized indices can be constructed that compare the performance of fund managers that swing for the fences, with those that aim for singles and doubles (to use a baseball analogy). All that is needed, is the performance period in the investor's memory, based on which the value of information $|\nu_0|$ is determined, in terms of deviation from normality.

3.1 Extreme Values of Combined Risk Tolerance

The coefficient ξ in (17) captures risk tolerance toward both variance and kurtosis. The value of ν, determined through indicator variable $I_{[\ ]}$ from any rolling sample, may push ξ toward infinity. In order to model such dynamics in this analysis, λ is set at 0.20 and $|\nu_0|$ is based on information from the past, related to the investment horizon. To the extent that kurtosis of portfolio returns in a rolling sample is different from $|\nu_0|$, the combined risk tolerance stays away from plus/minus infinity. The adjustment of weights in the second stage uses combinations of variance and kurtosis for which their product in tolerance ξ remains the same, based on λ and ν_0. Substitution of the sample variance into ξ leads to excess kurtosis that is consistent with a (λ, ν_0) pair. The investor thus sequentially adjusts quadratic utility weights to deviations from normality in sample *s*, as shown in (18) and (19). The process seeks evidence of kurtosis, which causes weights to change in preserving portfolio value. The formulation of tolerance to kurtosis in the process described involves comparing kurtosis in a specific sample, *s*, to the average kurtosis based in all previous samples. Combined risk tolerance, ξ_s, is conditional on the existence of kurtosis in the rolling sample *s*, relative to already processed rolling samples, *S*, within an overall time period. Thus, it is deviation from average rolling higher moments, and not normality in portfolio returns, which ultimately plays a role in adjusting portfolio weights.

Table 1. Extreme Values of Combined Risk Tolerance, ξ_s

Current sample kurtosis in relation to previous value:	Sample kurtosis approaches previous value from above	Sample kurtosis approaches previous value from below
Current sample portfolio returns are leptokurtotic	$\xi_s \to -\infty$	$\xi_s \to +\infty$
Current sample portfolio returns are platykurtotic	$\xi_s \to +\infty$	$\xi_s \to -\infty$

$$\mathbf{w}_s^T = \xi_s E[\mathbf{x}\cdot\mathbf{b}]_s^T \Omega_s^{-1} \tag{18}$$

$$\xi_s = \left[\frac{1}{\lambda} + \frac{\left(\theta_{\mathbf{q},s}^4 - 3\right)\sigma_{\mathbf{q},s}^2}{-I_{\left[\theta_{\mathbf{q},s}^4 > 3\right]}\left(\frac{\sum_{j=1}^{s}\left|\nu_{\mathbf{q},j}\right|}{s}\right) + I_{\left[\theta_{\mathbf{q},s}^4 < 3\right]}\left(\frac{\sum_{j=1}^{s}\left|\nu_{\mathbf{q},j}\right|}{s}\right)} \right]^{-1} \Rightarrow$$

$$\Rightarrow \xi_s = \lambda \left[1 - \frac{\theta_{\mathbf{q},s}^4 - 3}{\left\{I_{\left[\theta_{\mathbf{q},s}^4 > 3\right]} - I_{\left[\theta_{\mathbf{q},s}^4 < 3\right]}\right\}\left(\tilde{\theta}_{\mathbf{q},s}^4 - 3\right)} \right]^{-1} \tag{19}$$

3.2 Results of the Estimation Methodology

To achieve better results the investor undergoes a preference reversal in resolving the trade-off between two moments of portfolio return distribution. As soon as the portfolio returns deviate from normality, the investor recalculates weights, affecting profits in ways that depend on a combined tolerance toward both variance and kurtosis, which compares past experience (depending on investment horizons) to recent events. Thus, investors follow a smooth utility function, which is, ultimately, not strictly concave. The function is initially concave in returns, while deviations from portfolio normality trigger a reaction toward kurtosis in returns in a predictable manner, as shown in equation (4). The components above are treated as underlying factors, whose return is optimally weighted by the fixed-income portfolio manager.

$$E\left[U\left(r_p\right)\right] = \mathbf{q}^{\mathrm{T}} E\left(\mathbf{x}\cdot\mathbf{b}\right) - \frac{1}{2\cdot\lambda}\cdot\mathbf{q}^{\mathrm{T}}\Omega\,\mathbf{q} - \frac{\theta_p^4 - 3}{4\cdot\nu}\cdot\left(\mathbf{q}^{\mathrm{T}}\Omega\,\mathbf{q}\right)^2 \qquad (20)$$

In (20), $\mathbf{q}^T$ is a vector of portfolio weights, and $E(\boldsymbol{r}_p)$ is expected component returns that are products of the vector of daily yield changes $\mathbf{x} = [\Delta USyield_i, \Delta CNyield_i]$ times eigenvectors **b,** where *i* stands for the 3- and 6-month, as well as 1-, 2-, 3-, 5-, 7-, 10-, 20- and 30-year tenors of government bond yields, in the U.S. and Canada. Vector **b** contains eigenvectors for the $Level_t$, $Switch_t$, $Slope_t$, $Twist_t$ and $Curve_t$ on these changes in government yields. The rest five of the available eigenvectors are set aside, for parsimony. Also, $\sigma_p^2 = \mathbf{q}^T\mathbf{\Omega}\mathbf{q}$ is portfolio variance, and $\theta_p^4 - 3$ is the excess kurtosis of portfolio returns over that of a normal distribution. When the fourth moment of portfolio returns around the mean equals that of the normal distribution, that is, when $\theta_p^4 = 3$, that term in (4) disappears. Then, maximizing equation (4) with respect to **q** amounts to assuming mean-variance utility of wealth, with variance-risk tolerance λ and coefficient of risk aversion $-1/2\lambda$. Similar to variance-risk tolerance λ, there is tolerance toward kurtotic portfolio returns, ν. A-priori the institutional investor should associate leptokurtosis ($\theta_p^4 > 3$) with aversion to unexpected events, and platykurtosis to possible risk mitigation. Thus, kurtosis tolerance should be positive, as returns are usually leptokurtotic ($\nu > 0$, $\theta_p^4 > 3$). In that case, the coefficient $-(\theta_p^4 - 3)/4\nu$ reduces expected utility, depending on the squared covariance matrix in (4). Assuming that $\theta_p^4 = 3$ and utility quadratic, the investor derives weights from first-order condition (5). She then observes actual kurtosis in portfolio returns ($\theta_p^4 > 3$) and adjusts weights based on combined tolerance ξ, in (6). The second-order condition for maximization is that the product $-\xi^{-1}\Omega$ is negative semi-definite.

$$\left.\frac{\partial E\left[U\left(r_p\right)\right]}{\partial \mathbf{q}}\right|_{\theta_p^4=3} = 0 \Rightarrow \mathbf{q}^T = E\left(\mathbf{x}\cdot\mathbf{b}\right)^T\cdot\lambda\cdot\Omega^{-1} \qquad (21)$$

$$\left.\frac{\partial E\left[U\left(r_p\right)\right]}{\partial \mathbf{q}}\right|_{\theta_p^4>3} = 0 \Rightarrow \mathbf{q}^T = E\left(\mathbf{x}\cdot\mathbf{b}\right)^T\cdot\xi\cdot\Omega^{-1} \qquad (22)$$

$$\left.\frac{\partial^2 E\left[U\left(r_p\right)\right]}{\partial \mathbf{q}^2}\right|_{\theta_p^4>3} \le 0 \Rightarrow -\xi^{-1}\cdot\Omega \le 0 \quad \text{where}: \xi^{-1} \equiv \lambda^{-1} + \nu^{-1}E\left[\left(\theta_p^4 - 3\right)\sigma_p^2\right] \qquad (23)$$

The sufficient condition (23) allows for analysis of the behavior of investors who tolerate deviations from index on a variance basis, but are concerned with fat tails of the portfolio return distribution and may respond to kurtosis resulting from mean-variance optimization. If (23) is satisfied, then relations (8) are true, for the case of two assets (the n-asset case is similar). From the two relations in (8) I derive (9). The first set shows that tolerance ν can be negative in the case of leptokurtosis, as (θ_p^4 – 3), σ_p^2, λ are all positive, contradicting the claim that investors are always leptokurtosis-averse. The second set of relations in (8) and (9) holds when squared covariance between the factors is greater than the product of the each factor variance, which is more likely to occur if factors are correlated. Facing the prospect that both sets of relations hold simultaneously, the investor undergoes a 'preference reversal' that leads to $\nu = -(\theta_p^4 - 3)\ \sigma_p^2\ \lambda < 0$. The fact that ν is indeed negative proves that investors face a trade-off between risk coming from variance, and uncertainty that stems from leptokurtosis in portfolio returns. Kurtosis tolerance is negative and the term $-(\theta_p^4 - 3)\ /\ 4\nu$ increases, instead of decreasing expected utility. Preference reversal is accounted for.

$$|H_1| = -\left[\lambda^{-1} + \nu^{-1}\left(\theta_p^4 - 3\right)\sigma_p^2\right]\cdot|\sigma_{11}| \le 0;$$

$$|H_2| = -\left[\lambda^{-1} + \nu^{-1}\left(\theta_p^4 - 3\right)\sigma_p^2\right]\cdot\begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{vmatrix} \ge 0 \qquad (24)$$

$$|\sigma_{11}| > 0 \Rightarrow \nu \ge -\left(\theta_p^4 - 3\right)\cdot\sigma_p^2\cdot\lambda \ne 0;\ \text{if } \sigma_{11}\sigma_{22} \le \sigma_{12}^2 \text{ then}$$

$$\nu \le -\left(\theta_p^4 - 3\right)\cdot\sigma_p^2\cdot\lambda \ne 0 \qquad (25)$$

$$\text{iso-risk contour: } \left\{-\nu/\lambda\right\}_p^c = \left(\theta_p^4 - 3\right)\cdot\sigma_p^2 \equiv \pi_p^c \Rightarrow \left(\theta_p^4 - 3\right) = \pi_p^c/\sigma_p^2 \qquad (26)$$

$$\lim_{\pi_p^c \to \bar{\pi}_p} \xi = \lim_{\pi_p^c \to \bar{\pi}_p}\left\{\frac{1}{\lambda} - \frac{E\left[\left(\theta_p^4 - 3\right)\sigma_p^2\right]}{\left(\theta_p^4 - 3\right)\cdot\sigma_p^2\cdot\lambda}\right\}^{-1} = \lim_{\pi_p^c \to \bar{\pi}_p}\left[\frac{1}{\lambda} - \frac{\bar{\pi}_p}{\pi_p^c\cdot\lambda}\right]^{-1} = \infty \Rightarrow$$

$$\mathbf{q}^T = E\left(\mathbf{x}\cdot\mathbf{b}\right)^T\cdot\xi\cdot\Omega^{-1} \to \infty \qquad (27)$$

Equation (26) illustrates the iso-effect contour line between variance and kurtosis, for which the (negative of the) ratio of tolerances toward each one of them is equal to π_p^c, a constant. Interpreted differently, equation (26) shows the level of excess kurtosis ($\theta_p^4 - 3$) that an investor will tolerate for a level of variance σ_p^2, in order to remain on the indifference contour π_p^c, between the two portfolio distribution moments. In treatments of a quadratic portfolio optimization the tolerance toward variance λ is kept constant (I use $\lambda = 0.20$). Preference reversal associated with ν must occur instantaneously, while the product $E[(\theta_p^4 - 3)\cdot\sigma_p^2]$ in the combined risk tolerance ξ represents the average experience of the investor. As the 'experienced' manager bases estimates of $E[(\theta_p^4 - 3)\cdot\sigma_p^2]$ on a sample larger than that for $(\theta_p^4 - 3)$ σ_p^2 in ν, the portfolio weights $\mathbf{q}^T = E(\mathbf{x}\cdot\mathbf{b})^T\,\xi\,\Omega^{-1}$ do not become infinite, as shown in equation (27). Implications of this infinity include the fact that some wealth management professionals, in the absence of concrete tracking error or other constraints, will tend to change portfolio weights abruptly in order to take advantage of uncertain characteristics that persist, which may be detrimental to other investors. The source of the uncertainty, may originate from premises that today we associate with the financial crisis of 2007 – 2008.

3.3 Kurtosis Tolerance and Probability of Outperformance

Several facets of retail and institutional money management are of dire concern to investors, in the current market environment of low interest rates, and subdued volatility. The presumption, or fact, that current yields do not capture both linear and non-linear risk fuels investor interest in strategies that have new names, such as 'unconstrained,' 'buy-and-hold', or 'smart beta.' In institutional investing, sponsors have become increasingly aware that matching liabilities with investments that have minimal tail risk is not feasible. Thus, the right mix of a tolerance toward abrupt market events, versus that toward normal volatility, needed to be precisely defined. The level of on-going discussion about such risk preferences amounts to attempts at exhuming out of a plan sponsor, a vaguely quantifiable tolerance toward risk. Based on the iso-risk contour methodology described above, assigning a numerical value to such tolerance is straightforward. The investor seeks a trade-off between returns from two kinds of exposure: normal volatility and regime-switching, kurtosis-generating tails. The ratio of risk tolerances, ν/λ, is the main determinant of the risk appetite (and concomitant returns) as opposed to one or the other, in isolation.

$$\left(\theta_{\mathrm{p}}^4 - 3\right)^e \cdot \sigma_p^2 = -\frac{\nu}{\lambda} \Rightarrow \nu = -\lambda \cdot \left(\theta_{\mathrm{p}}^4 - 3\right)^e \cdot \sigma_p^2 < 0 \qquad (28)$$

For a fixed value of tolerance toward variance-risk, λ, the required tolerance of the investor toward fat-tail events, ν, becomes a function of some long-term estimate of expected kurtosis $(\theta_p^4 - 3)^e$, times variance in strategy returns. In the usual case of leptokurtosis, tolerance ν is negative, indicating the degree to which investors undergo a preference reversal toward seeking returns from market events. In the case of liability-matching, it is often the case that a large portion of the portfolio is generated from exposure to normal market events, while the remainder is invested in a way that generates returns when markets undergo some abrupt change, in general. In the recent environment of low interest rates and rate volatility, this latter portion leads to the necessity for justifying a level of tolerance toward abrupt market events. This tolerance is quantified in (28). For this analysis, tolerance toward market events was estimated for different kinds of fixed income strategies, which pertain to funds in a number of investable universes. Data is obtained from Bloomberg in estimating these tolerances, based on the methodology above. In the process, I attempt to link the performance of such strategies to the tolerance toward fat tails ν, estimated.

In addition, results show that matching the actual probability of outperformance for a number of fixed income strategies to the tolerance for fat-tail risk of the investor, validates one of the main constructs of Prospect Theory, as that is generally stated in Kahneman and Tversky's (1979) 'certainty effect.' Prospect Theory, in general, states that investors become risk-seeking when wealth diminishes and risk averse when it increases. In the context of iso-risk contours, investors optimally change from kurtosis-averse to kurtosis-seeking, depending on distributional characteristics of market returns. Through the use of an actual performance measurement method for fixed-income strategies, I show that when the probability of outperforming a fitting benchmark is less than 50%, kurtosis-seeking behavior becomes more pronounced, and the opposite. The preference reversal, captured by a change in sign of kurtosis-tolerance ν, remains; it just becomes more negative (investors are more kurtosis-seeking), if the probability of outperformance is less than half, and less negative (investors re less kurtosis-seeking) if that probability is greater than one half, approximately. In a general sense, iso-risk contours support the findings of Prospect Theory. This finding is arrived at through logistic regression of strategy attributes. Generally, it is assumed that the probability distribution of several fixed income portfolio managers depends on the sensitivity of their managed strategies to features of the yield curve, such as level, slope and curvature. Returns of roughly 500 fixed income strategies are regressed against four principal components of the swap curve. These components are derived through the orthogonal decomposition of twelve points on the swap curve. These twelve points are lagged once; and once again for a total of thirty-six independent variables. The resulting four components of highest eigenvalue capture the phenomena of (i) duration: an almost-parallel shift in the curve, (ii) a flattening of the curve, commonly associated with monetary policy, (iii) a 'liquidity-trap' kind of effect, in which abrupt changes in rates from one month to the next have little effect, and finally, (iv) quantitative easing or tapering, which directly impacts the mid-section of the swap curve. These effects arise from orthogonal decomposition of these thirty six variables. The sensitivities of strategy i to each of j = 2… 5 independent variables (principal components, PC_j) are:

$$E[\mathrm{R_i}] = \alpha_i + \sum_{j=1}^{4} \beta_{i,j} \cdot PC_j \qquad (29)$$

To remove arbitrariness in fitting an index to a fund, a similar equation is estimated for the benchmark, in which alpha is restricted to zero. Thus the information ratio for each strategy is estimated. The cumulative distribution of all information ratios for all strategies represents the probability of (out)-performance, for each strategy P_j. The logistic regression model uses this probability of outperformance, against what can now be considered as attributes of each strategy i, namely, α_i and $\beta_{i,j}$. Probability of performance rests on these attributes:

$$ln\frac{\mathrm{P_i}}{1\text{-}\mathrm{P_i}} = b_{0,i} + b_{1,i}\cdot\alpha_i + \sum_{j=2}^{5} b_{j,i}\cdot\beta_{i,j} + \varepsilon_i \qquad (30)$$

The above relation is one usual method of assigning probability of performance to strategy *i*, on sensitivities of the strategy to movements in the yield curve, in the case of fixed income portfolios. This regression is run across all strategies at the same time. At the same time, it is shown that the level of iso-risk (the product of excess kurtosis and variance) is strongly related to same attributes of portfolio strategies, α_i and $\beta_{i,j}$ for j=2….5, for equation (28). Linearization of this product leads to the following estimation of iso-risk, based on the same strategy attributes:

$$\ln\left(\theta_{\mathrm{p}}^4 - 3\right)_i + \ln\left(\sigma_p^2\right) = b_{0,i} + b_{1,i}\cdot\alpha_i + \sum_{j=2}^{5} b_{j,i}\cdot\beta_{i,j} + \varepsilon_i \qquad (31)$$

The challenge is to relate the probability of outperformance, above, with the level of iso-risk taken. One way to establish such relation is to relate the coefficients $b_{0,i}$, $b_{1,i}$ and $b_{j,i}$ to $c_{0,i}$, $c_{1,i}$ and $c_{j,I}$ for j=2….5 and across strategies i = 1….500 across the whole time period of the sample, which includes the 2008 financial crisis. The pivotal finding of the present analysis is that this regression has very high statistical significance. The regression of the second set of coefficients against the first results in the sensitivity of outperformance probability to iso-risk, δ, which was found to be remarkably stable, against all strategies considered. Equation (33) links the iso-risk contour π^e to the probability of outperformance *P*, for each strategy in the sample. It is easy to estimate the effect of outperformance probability, on the investor tolerance toward events, ν, by substituting an earlier relation into equation (34). Importantly, γ in (33) denotes idiosyncratic (non-performance-related) aspects of a fund that has zero alpha ($\alpha = 0$) and is self-financing ($\Sigma\beta_{\mathrm{i,j}} = 0$).

$$c_{0,i} = \gamma + \delta\cdot\beta_{0,i}, c_{1,i} = \gamma + \delta\cdot\beta_{1,i} \text{ and } c_{j,i} = \gamma + \delta\cdot\beta_{j,i} \text{ for } \mathrm{j} = 2,...,5 \qquad (32)$$

$$\ln\left(\theta_p^4 - 3\right)_i + \ln\left(\sigma_p^2\right)_i = \gamma\left(1 + \alpha + \sum_{j=2}^{5}\beta_{i,j}\right) + \delta\left(\ln\frac{P_i}{1 - P_i}\right) + \zeta_i$$

$$\pi^e = \left(\theta_p^4 - 3\right)^e\cdot\left(\sigma_p^2\right) = \eta\cdot\frac{P}{1-P}, \text{ where}: \eta = \exp\left[\gamma\left(1 + \alpha + \sum_{\mathrm{j=2}}^{5}\beta_{\mathrm{i,j}}\right)\right] \qquad (33)$$

$$\nu = -\lambda\cdot\left(\theta_p^4 - 3\right)^e\cdot\left(\sigma_p^2\right) = -\lambda\left[\eta\left(\frac{P}{1-P}\right)^{\delta}\right] \qquad (34)$$

$$\frac{\partial \nu}{\partial P} = -\lambda\eta\delta\left(\frac{P}{1-P}\right)^{\delta-1} \cdot \frac{(1-2P)}{(1-P)^2} < 0, \text{if } P < 0.5 \qquad (35)$$

The sign of the above derivative clearly depends on the term [1 - 2*P*] in square brackets, above. If δ is positive (already estimated at around 2.5 in this work) and the probability of outperformance is less than one half ($P < 0.5$ and thus $1 - 2P > 0$), the above derivative is negative. Given that ν is negative in sign, due to preference reversal, explained above, the fact that outperformance probability is less than half makes this reversal even accentuated, promulgating investors to seek further returns from abrupt market events. The opposite of true, if the probability of outperformance is less than one half. In that case, the above derivative is positive, which means that the preference reversal toward fat-tailed events becomes partially retarded. The actual value of the accentuation or retardation effect of outperformance probability on the tolerance toward kurtosis depends, among other things, on the value of δ, which is estimated in this analysis for a number of fixed income strategy universes.

In the above relation, it is assumed that all of the variables λ, η and δ, are positive. Specifically, λ can be given a positive value (such as 0.20) to signify that investors are always averse to variance-risk, while δ is the result of estimation, preliminary values for which range around 2.5, in the sample of five hundred fixed income strategies, examined. The interpretation of the η variable leads to differences in changes in tolerance toward kurtosis, due to probability of outperformance. Cases below pertain to different types of strategy assumptions, such as (i) fully invested, (ii) index-tracking (zero-alpha) and (iii) long-short. At the outset, it is interesting to note that restricting γ to zero results in $\eta = 1$, and eliminates this variable.

3.4 Restatement of Sharpe Ratio and Information Ratio

The investor has to be comparing deviations from normality in the recent period, from those across some long-term path, as discussed above. In the section where the rolling ξ was derived, it was found that deviations from long-term deviations, and not from normality, matter. That discussion of ξ above signifies that performance measurement is best gauged by restating performance ratios as follows:

$$SharpeRatio = \frac{E\left[R_p - R_f\right]}{\left\{\left[\frac{\left(\theta_p^4 - 3\right)\cdot\sigma_p^2}{\left(\theta_p^4 - 3\right)^e}\right]_{R_p - R_f}\right\}^{0.5}}, \text{ and } \lim_{\theta_p^4 \to 3}\left[\frac{\left(\theta_p^4 - 3\right)\cdot\sigma_p^2}{\left(\theta_p^4 - 3\right)^e}\right]_{R_p - R_f} = \left[\sigma_p^2\right]_{R_p - R_f}$$

$$InformationRatio = \frac{E\left[R_p - R_b\right]}{\left\{\left[\frac{\left(\theta_p^4 - 3\right)\cdot\sigma_p^2}{\left(\theta_p^4 - 3\right)^e}\right]_{R_p - R_b}\right\}^{0.5}},\ \text{and} \lim_{\theta_p^4 \to 3}\left[\frac{\left(\theta_p^4 - 3\right)\cdot\sigma_p^2}{\left(\theta_p^4 - 3\right)^e}\right]_{R_p - R_b} = \left[\sigma_p^2\right]_{R_p - R_b}$$

4. Conclusion

In this analysis, the mathematical mechanics of the second and fourth moments of the distribution of portfolio returns help derive conditions that pertain to patterns of investor behavior during the recent 2007-2008 financial crisis. These conditions rest on a simple extension of the classic quadratic utility optimization that treats kurtosis in a manner analogous to variance. The behavior of tolerance to kurtosis is examined through a two-stage utility maximization process, which is observed at time periods before, during, and after the crisis. The polynomial utility becomes quadratic in portfolio return variance. Thus, it possesses a minimum beyond which the ‘prudent’ investor is not shown to expand. Instead, he operates on an iso-ratio relation between kurtosis and risk tolerance, defined as the product of excess kurtosis with portfolio variance. This ratio-indifference isolates instances when the investor either responds abruptly to consistent kurtosis in portfolio returns, or remains vigilant as kurtosis does not exceed entrenched values. One way or another, the investor manages to maintain portfolio value in the face of portfolio kurtosis, through the recent financial crisis. This hypothetical investor appears to have sustained the crisis just fine.

One should be expecting rougher times, as forces that made the markets bounce against liquidity constraints in the past may today be pushing and pulling investors in and out of their learnt reactions, leading to high portfolio losses. Investors stand at least as good a chance to lose value when markets revert back to a post-crisis state, as they did at the start of the crisis. From a practical standpoint, this observation makes sense. The liquidity-drained financial system in the middle of the crisis has engrained in it the abrupt asset responses and portfolio kurtosis, leading to liquidity issues that further accentuate asset responses. With kurtosis present, the investor remains vigilant in adjusting portfolio weights in a way that mitigates the large return variance. If the economy reverts to a pre-crisis state, the large variance remains. With kurtosis absent, the secondary adjustment is not triggered. The new, unmitigated variance will likely reduce both investor utility and portfolio value.

Notes

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[2] The estimation of carry trade returns was presented elsewhere (see Xanthopoulos, Apostolos (2008), “Nonlinearity and the Forward Premium Anomaly”, *Journal of Business and Economics Research*, 6 (7), 113-127). The future spot rate responds to linear and nonlinear rate effects. The expectation formulation for spot rate contains a hyperbolic-tangent response to interest rate differentials, which justifies kurtosis.
[3] The axiomatic characteristics of Von Neumann-Morgenstern cardinal utility (VNM) are not explored. This study follows the perspective of illustrating problems with standard utility theory through a model. The underlying focus of the study is to develop and test a programmable application, which determines optimal weights.
[4] For example, in Brunnermeier (2008), carry trades are subject to crash risk due to the sudden unwinding that occurs when risk appetite and funding liquidity change. In the present analysis, investors hedge some of the crash risk by manipulating ‘normal’ volatility. This finding is reflected in the estimated asset return models, and then modeled in the utility function that is proposed and applied to portfolio returns.
[5] This method has been presented to and reviewed by money management consulting firms in the Chicago area. The treatment of the hedging behavior between variance and kurtosis concurred with a wide experience in the portfolio management arena.
[6] Optimizing expected utility in (4) with respect to weights **q** requires the investor to invert a co-kurtosis matrix, $(\mathbf{q}^T\Omega\mathbf{q})^2$. This task becomes unnecessary in this analysis. Also, constrained utility is optimized programmatically. Nevertheless, the problem is restated so that the combined tolerance toward both moments of the portfolio return distribution follows from optimizing weights in (4), in unconstrained maximization.

Appendix A. Kurtosis in Rolling Samples

This study uses seventy rolling samples, in each period. For each sub-sample, **q-** and **w**-weights lead to returns from quadratic and polynomial utility. A reasonable assumption is that the investor incorporates past information about variance/kurtosis combinations with individual weights across time. In this analysis, information is weighted equally across time, into $|\nu_0|$. As the calculation progresses from rolling sample 1 to rolling sample s, information about kurtosis and its investor tolerance is thus accumulated. This average absolute value of kurtosis tolerance up to rolling sample s, $E|\nu_{\mathbf{q},\mathbf{s}}|$, is defined below and corresponds to $|\nu_0|$ in (17). This running average of kurtosis tolerance determines the combined tolerance ξ_s in each sub-sample s, which in turn determines the polynomial weights $\mathbf{w}_s$. Thus, the deviation from normality prompts the speculator to adjust quadratic weights in current sample s, using $\mathbf{w}_s^T = \xi_s\, E[\mathbf{xb}]_s^T \Omega_s^{-1}$, which depends on the combined risk tolerance, ξ_s.

$$|\nu_0| \equiv E|\nu_{\mathbf{q},s}| = \frac{\sum_{j=1}^{s} |\nu_{\mathbf{q},j}|}{s} \quad ; \qquad \xi_s = \left[\frac{1}{\lambda} + \frac{(\theta_{\mathbf{q},s}^4 - 3)\sigma_{\mathbf{q},s}^2}{-I_{[\theta_{\mathbf{q},s}^4 > 3]} \cdot E|\nu_{\mathbf{q},s}| + I_{[\theta_{\mathbf{q},s}^4 < 3]} \cdot E|\nu_{\mathbf{q},s}|} \right]^{-1}$$

$$\mathbf{w}_s^T = \xi_s \, E[\mathbf{x}\cdot\mathbf{b}]_s^T \cdot \Omega_s^{-1}$$

The average absolute value of the product of portfolio variance and kurtosis (average $|-\pi_{\mathbf{q},s}|$) defines an iso-ratio curve of the two tolerances. Based on equation (11) above, $|-\pi_{\mathbf{q},s}|=|-\nu_{\mathbf{q},s}/\lambda|$ takes the sign of $-\nu_{\mathbf{q},s}$, as λ=20%>0 remains fixed (the investor is variance-averse). The accumulated product of kurtosis and variance for sub-samples j is $\pi_{\mathbf{q},j}=(\theta_{\mathbf{q},j}^{4} - 3)\sigma_{\mathbf{q},j}^{2}$. The product for the current sample s is $\pi_{\mathbf{q},s}=(\theta_{\mathbf{q},s}^{4} - 3)\sigma_{\mathbf{q},s}^{2}$. This product is used in the running sub-sample average $|-\pi_{\mathbf{q},s}|$, which is factored-out of the denominator of the definition of ξ, as shown below. Based on $\mathbf{q}_s$ and $(\sigma_{\mathbf{q},2})^2$ in sub-sample s the kurtosis corresponding to $E|-\pi_{\mathbf{q},s}|$ from previous rolling samples is applied in estimating risk tolerance ξ_s. Depending on whether kurtosis in sample s approaches this benchmark from above or below, the combined risk tolerance either becomes infinite in risk aversion or alters speculator behavior to risk-seeking.

The analysis is also re-coined in terms of accumulated kurtosis in rolling samples. Deviations $(\theta_{\mathbf{q},s})^4$ from normality in rolling sample, s, are compared against accumulated deviations from normality, $[(\theta_{\mathbf{q},s})^4$~tilde]. Once again, it is the deviation from average kurtosis that affects kurtosis tolerance, and not normality in returns.

$$\xi_s = \lambda\left[1-\frac{\pi_{\mathbf{q},s}}{\left\{I_{[\theta_{\mathbf{q},s}^4>3]}-I_{[\theta_{\mathbf{q},s}^4>3]}\right\}\left|-\bar{\pi}_{\mathbf{q},s}\right|}\right]^{-1} \quad \text{where} \quad \left|-\bar{\pi}_{\mathbf{q},s}\right| \equiv \frac{\sum_{j=1}^{s}\left|-\pi_{\mathbf{q},j}\right|}{s} = E\left|-\pi_{\mathbf{q},s}\right|$$

Barring any immediate and infinite weight changes at the prudent point of his combined risk tolerance, the investor compares information about rolling sample s, to that of the accumulated kurtosis in all previous samples, as shown below. This processing of information on portfolio return kurtosis is reflected in the following equations and is summarized in Table 1, above.

$$\left(\tilde{\theta}_{\mathbf{q},s}^{4}-3\right)\sigma_{\mathbf{q},s}^{2} = \left|\bar{\pi}_{\mathbf{q},s}\right| \Rightarrow \left(\tilde{\theta}_{\mathbf{q},s}^{4}-3\right) = \frac{\left|\bar{\pi}_{\mathbf{q},s}\right|}{\sigma_{\mathbf{q},s}^{2}} > 0 \,.$$

$$\xi_s = \lambda\left[1 - \frac{\pi_{\mathbf{q},s} \big/ \sigma^2_{\mathbf{q},s}}{\left\{I_{\left[\theta^4_{\mathbf{q},s}>3\right]} - I_{\left[\theta^4_{\mathbf{q},s}>3\right]}\right\}\left|\bar{\pi}_{\mathbf{q},s}\right| \big/ \sigma^2_{\mathbf{q},s}}\right]^{-1} = \lambda\left[1 - \frac{\overset{4}{\theta}_{\mathbf{q},s} - 3}{\left\{I_{\left[\theta^4_{\mathbf{q},s}>3\right]} - I_{\left[\theta^4_{\mathbf{q},s}>3\right]}\right\}\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)}\right]^{-1}$$

If portfolio returns from quadratic utility based on a current sample are leptokurtic, the indicator value for kurtosis tolerance is positive. As current-sample kurtosis approaches that in previous samples from above (below), combined risk-tolerance and its impact on weights goes to minus (plus) infinity.

$$\xi_s = \lambda\left[1 - \frac{\overset{4}{\theta}_{\mathbf{q},s} - 3}{\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)}\right]^{-1} \Rightarrow \lim_{\theta^4_{\mathbf{q},s} - 3 \to \left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)^{(\pm)}} \xi_s = \lambda\left[1 - \frac{\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right) \pm o}{\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)}\right]^{-1} = \mp\infty$$

If portfolio returns from quadratic utility based on a current sample are platykurtotic, the indicator value of kurtosis tolerance is negative. As current-sample kurtosis approaches that in previous samples from above (below), combined risk-tolerance and its impact on weights goes to plus (minus) infinity.

$$\xi_s = \lambda\left[1 - \frac{\overset{4}{\theta}_{\mathbf{q},s} - 3}{-\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)}\right]^{-1} \Rightarrow \lim_{\theta^4_{\mathbf{q},s} - 3 \to -\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)^{(\pm)}} \xi_s = \lambda\left[1 - \frac{-\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right) \pm o}{-\left(\overset{\sim 4}{\theta}_{\mathbf{q},s} - 3\right)}\right]^{-1} = \pm\infty$$

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